Signed posets and a B-symmetric generalization of Stanley's acyclicity theorem

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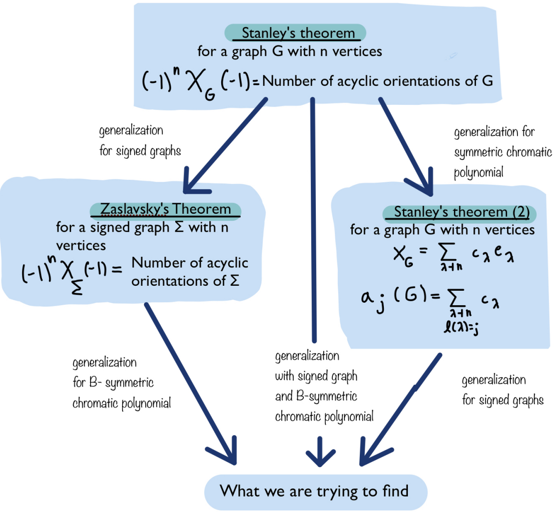
joint work with Eric Fawcett, Torey Hilbert and Mikey Reilly under Sergei Chmutov

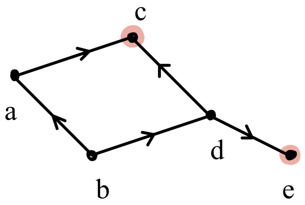
The Ohio State University

August 16, 2020

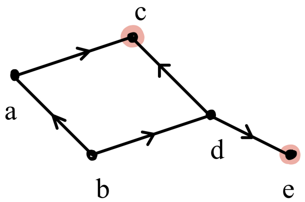
Our Goal


Stanley's theorem
for a graph G with n vertices
$(-1)^n \chi_G(-1) =$ Number of acyclic orientations of G
generalization for signed graphs
Zaslavsky's Theorem
for a signed graph Σ with n vertices
$(-1)^n \chi_\Sigma(-1) =$ Number of acyclic orientations of Σ
generalization for symmetric chromatic polynomial
Stanley's theorem (2)
for a graph G with n vertices
$X_G = \sum_{\lambda \vdash n} c_\lambda e_\lambda$
$a_j(G) = \sum_{\substack{\lambda \vdash n \\ \ell(\lambda)=j}} c_\lambda$
generalization for B- symmetric chromatic polynomial
generalization with signed graph and B-symmetric chromatic polynomial
generalization for signed graphs
What we are trying to find

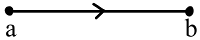
Graphs and Posets



Graphs and Posets



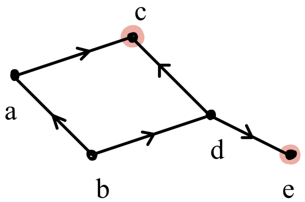
Arrow Convention:



Poset: A partially ordered set

$$b < a < c$$
$$b < d < c$$
$$b < d < e$$

Graphs and Posets



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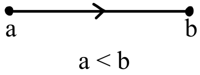

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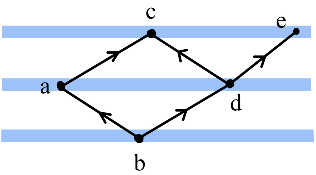
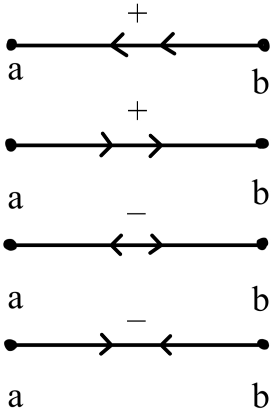
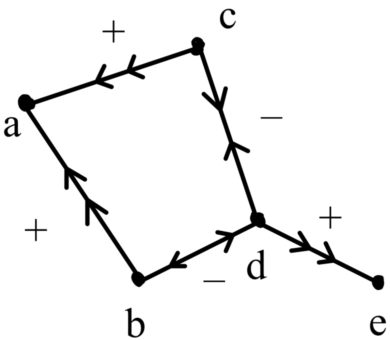


Signed Graphs

Possible Edges:



Signed Graph



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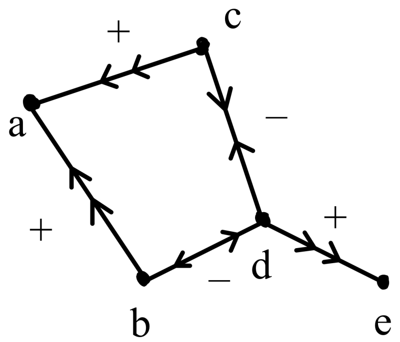
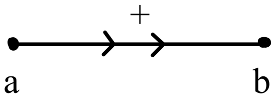


How can we tell if it's acyclic?

Covering Graphs

Signed Graph:



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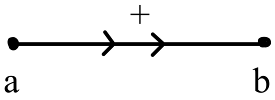

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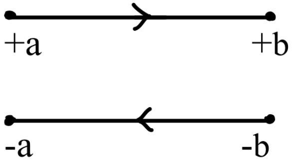

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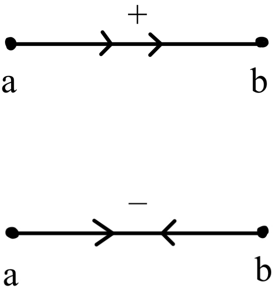

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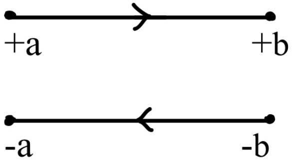

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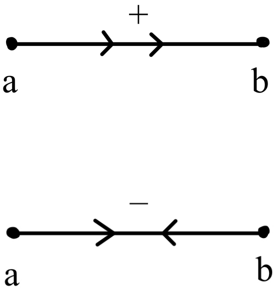

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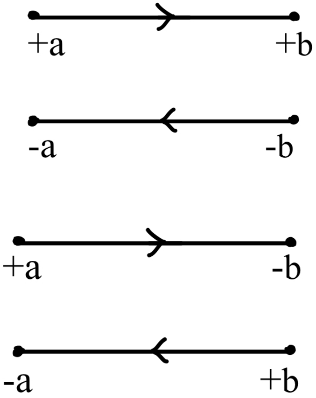
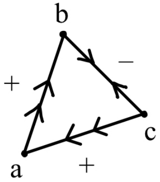


Covering Graph Examples

Signed Graph:



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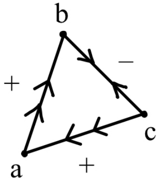

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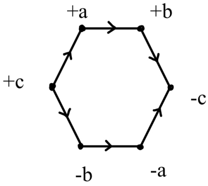

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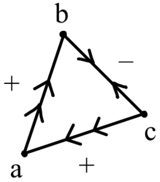

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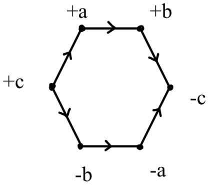

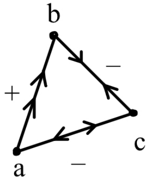

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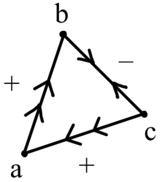

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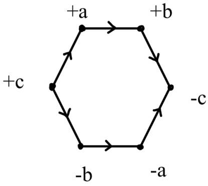

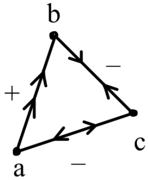

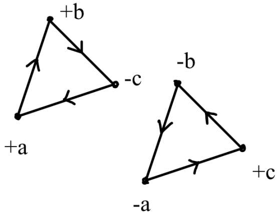

Signed Posets

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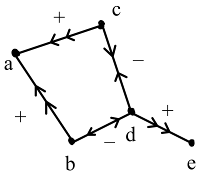

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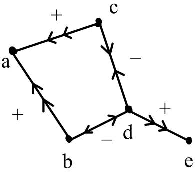

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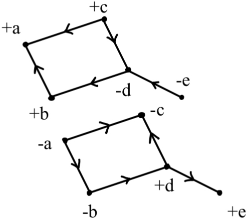

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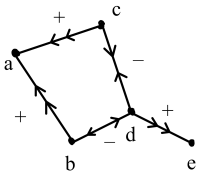

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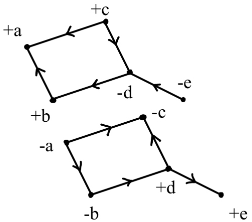

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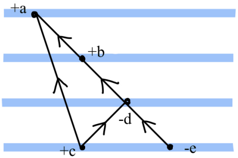

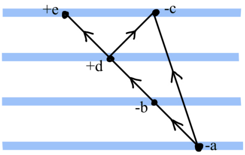

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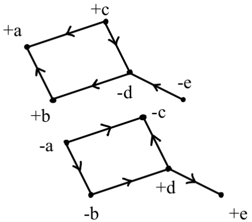

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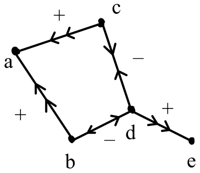

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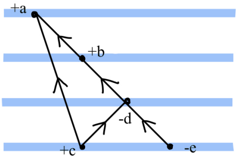

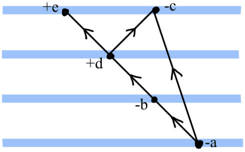

$+c < -d < +b < +a$

$-d > -e$

$-c > +d > -b > -a$

$+d < +e$

Proper coloring

Proper coloring: A function $\kappa \colon V(G) \to \mathbb{Z}$ such that for all adjacent vertices v, w, $\kappa(v) \neq \sigma(v, w)\kappa(w)$.

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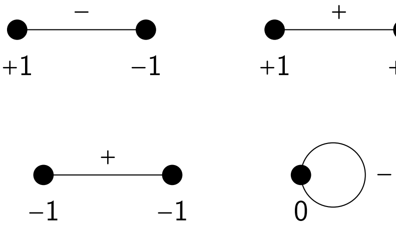
Examples of improper coloring:



B-symmetric chromatic function

$$Y_G(\dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots) := \sum_{\substack{\kappa : V(G) \to \mathbb{Z} \\ \text{proper}}} \prod_{v \in V(G)} x_{\kappa(v)}$$

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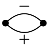
Examples:

Y 

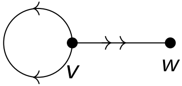
$$= \cdots + x_{-2} + x_{-1} + x_1 + x_2 + \cdots$$

Y 

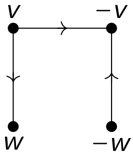
$$= \sum_{i,j} x_i x_j - \sum_i x_i^2 - 2\sum_i x_i x_{-i} + 2x_0^2$$

Linear extension of a signed poset

Poset P**:**



Lift $\tilde{P}$**:**



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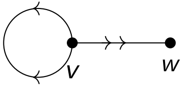

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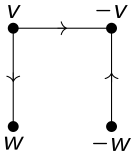
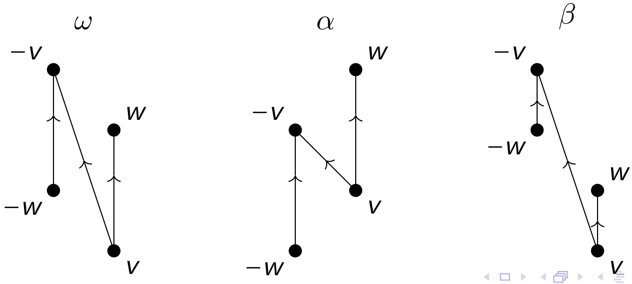


B-symmetric linear extensions:



Order-preserving coloring

If $v < w$ in the lift $\tilde{P}$ and $\kappa\colon \tilde{P} \to \mathbb{Z}$ is **order-preserving**, then $\kappa(v) < \kappa(w)$. **Note:** $\kappa(v) = -\kappa(-v)$

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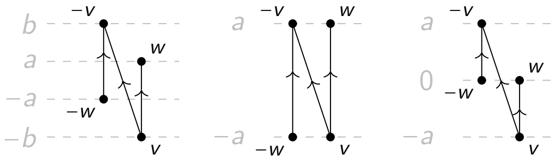
Some examples:



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Specifically, we will find a linear map $\varphi\colon \mathrm{BSym} \to \mathbb{Z}[t]$ such that

$$\varphi(Y_G) = \sum_{k=0}^{\infty} \mathrm{sink}_G(k) t^k.$$

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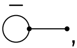
For example, if $G =$ , then $\varphi(Y_G) = 1 + 3t$:

Step-by-step

Step 1: Decompose Y_G into a sum, over signed posets, of quasi-B-symmetric functions:

$$Y_G = \sum_{\substack{P \text{ is an acyclic} \\ \text{orientation of } G}} Y_P$$

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Step 2: Find a convenient expression for Y_P as a sum over the linear extensions of P:

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Step 3: Use the convenient expression to find a linear map $\varphi\colon \mathrm{QBSym} \to \mathbb{Z}[t]$ such that for any signed poset P with k sinks, $\varphi(Y_P) = t^k$.

Step 1: Y_P

Given a signed poset P with vertices $v_1, \ldots, v_n$, define

$$Y_P = \sum_{\substack{\kappa \text{ is an order-preserving} \\ \text{coloring of } P}} x_{\kappa(v_1)} \cdots x_{\kappa(v_n)}.$$

Y_P is quasi-B-symmetric.

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Lemma

For any signed graph G,

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Lemma

For any signed graph G,

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Proof.

Given a coloring, induce the unique acyclic orientation of G which makes the coloring order-preserving. □

Step 2: A convenient expression

Our expression for Y_P is a sum over all order-preserving colorings, but we want to write it as a (finite) sum over just the linear extensions. But how?

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Compare linear extensions to encode how they can be "averaged" to give order-preserving colorings.

Step 2: A convenient expression, continued

Fix a linear extension ω. Given a linear extension α, how do we determine what α should contribute to Y_P?

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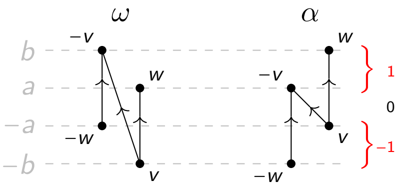
Look at the disagreements between two linear extensions:



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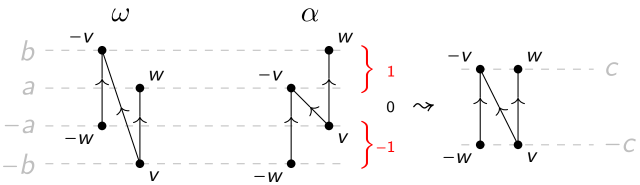

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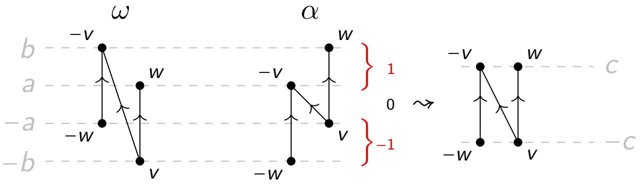

$$\sum_{0<a<b} x_{-a}x_b + \sum_{0<c} x_{-c}x_c$$

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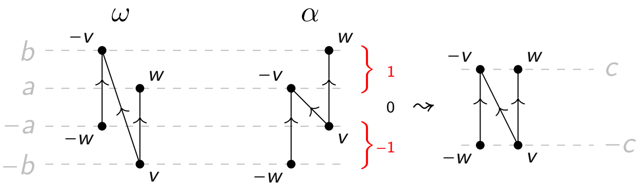


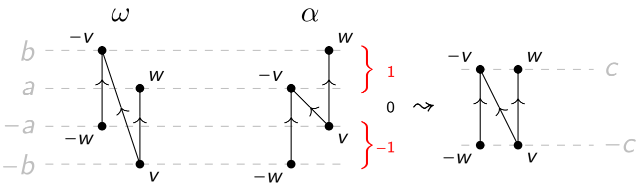
$$\sum_{0<a<b} x_{-a}x_b + \sum_{0<c} x_{-c}x_c$$

$$= \sum_{0<a\leq b} x_{-a}x_b$$

Step 2: A convenient expression, continued

Fix a linear extension ω. Given a linear extension α, how do we determine what α should contribute to Y_P?

Look at the disagreements between two linear extensions:



$$\sum_{0<a<b} x_{-a}x_b + \sum_{0<c} x_{-c}x_c$$

$$= \sum_{0<a\leq b} x_{-a}x_b$$

$$= Q_{\{0\},-+}$$

Step 2: A convenient expression, continued

For $S \subseteq \{0, \ldots, n-1\}$ and $\varepsilon \in \{-1, 1\}^n$, define

$$Q_{S,\varepsilon} := \sum_{\substack{0 \le i_1 \le \cdots \le i_n \\ s \in S \implies i_s < i_{s+1} \\ 0 \in S \implies 0 < i_1}} x_{\varepsilon_1 i_1} \cdots x_{\varepsilon_n i_n}.$$

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Lemma

Let P be a signed poset. Then

$$Y_P = \sum_{\substack{\alpha \text{ is a linear} \\ \text{extension of } P}} Q_{A(\alpha,\omega),\varepsilon(\alpha)}.$$

Step 3: An awful function

Let $S \subseteq \{0, \ldots, n-1\}$ and $\varepsilon \in \{-1, 1\}^n$. Then

$\varphi(Q_{S,\varepsilon}) :=$

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$$\varphi(Q_{S,\varepsilon}) := \begin{cases} t(t-1)^k & S = \{0, \ldots, n-k-1\} \text{ and } \\ & \varepsilon_i > 0 \text{ for each } i \in \{n-k, \ldots, n\} \\ (t-1)^k & S = \{0, \ldots, n-k-1\},\ \varepsilon_{n-k} < 0, \text{ and } \\ & \varepsilon_i > 0 \text{ for each } i \in \{n-k+1, \ldots, n\} \\ (t-1)^n & S = \varnothing \text{ and } \varepsilon_i > 0 \text{ for each } i \in \{1, \ldots, n\} \\ 0 & \text{otherwise.} \end{cases}$$

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Obnoxious obstruction: The $Q_{S,\varepsilon}$'s aren't linearly independent!

$$Q_{\{0\},-+} - Q_{\{0,1\},-+} = Q_{\{0\},+-} - Q_{\{0,1\},+-}.$$

The conclusion

Theorem

For any signed graph G,

$$\varphi(Y_G) = \sum_{k=0}^{\infty} \mathrm{sink}_G(k) t^k.$$

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Proof.

We have

$$\varphi(Y_G) = \sum_{\substack{Y \text{ is an acyclic} \\ \text{orientation of } G}} \varphi(Y_P)$$

$$= \sum_{\substack{Y \text{ is an acyclic} \\ \text{orientation of } G}} t^{\text{sink}(P)}$$

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What other nice properties does Y_G have?

What variations on Y_G might have nice properties?

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